Asset Allocation, Longevity Risk, Annuitisation and Bequests

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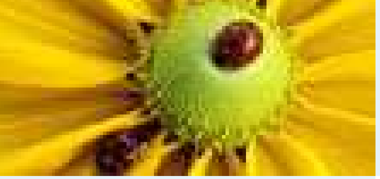
Outline

- Motivation
- Preliminaries
- Optimisation Problem
- Results
- Conclusions

Importance of the End of the Life-Cycle

- Rising Conditional Life Expectancies
- Growing Number of DC Plans
- Continuing Wealth Concentration Among Pensioners
- Input for Labour Models



Technical View of the Pensioner's Problem

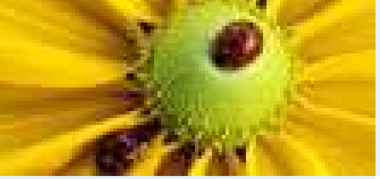
- Consumption/Portfolio Optimisation (c, π)
  $\longrightarrow$ Financial Market Risk
- Optimal Annuitisation Decision (τ)
  $\longrightarrow$ Longevity Risk

$\Rightarrow$ Combined Optimal Stopping and Optimal Control Problem (COSOCP)

Literature Overview

- Classical Literature
  - Merton (1969) $\longrightarrow$ Stochastic Control
  - Vast Literature Imposing a Fixed or Infinite Planning Horizon
  - Yaari (1965) $\longrightarrow$ Uncertain Lifetime
  - Richard (1975) $\longrightarrow$ Reversible Annuities
- Few Normative Models with Irreversible Annuities and Uncertain Lifetime, i.e.
  - Stabile (2006):
    Parsimonous Continuous-Time Model
    Annuitisation Rule as a Controlled Stopping Time
  - Milevsky and Young (2007):
    Focus on Mortality Law and Annuitisation Schemes

  $\Longrightarrow$ Both Papers: No Bequest Motive



Extensions to the Model of Stabile (2006)

- Inclusion of a Bequest Motive
- Prior Life Insurance and Subsistence Level of Bequest
- Economically Relevant Risk Aversion ($\gamma > 1$)
- New Solution Method with Duality Arguments

Main Model Assumptions

- Utility Maximisation (Consumption, Annuity, Bequest; Identical Relative Risk Aversion)
- No Stochastic Income
  $\longrightarrow$ No Labour Income
- Prior Decision on Annuitisation and Life Insurance Taken as Given
- Annuitisation of Entire Wealth and Consumption of Entire Annuity
  $\longrightarrow$ All-or-Nothing Framework
- Irreversible Annuitisation Decision
- One Riskless Asset, One Risky Asset (Geometric Brownian Motion)
- Exponential Mortality Law

Model Basics

- Life Expectancy and Annuity:

$$E^S[T_x] = \int_0^\infty \left({}_tp_x^S\right) dt = \int_0^\infty e^{-\lambda_x^S t} dt = \frac{1}{\lambda_x^S} \text{ and } E^O[T_x] = \frac{1}{\lambda_x^O}$$

$$\bar{a}_x = \int_0^\infty e^{-rt} \left({}_tp_x^O\right) dt = \int_0^\infty e^{-rt} e^{-\lambda_x^O t} dt = \frac{1}{r + \lambda_x^O}$$

- Control Variables and State Evolution:
  $\mathcal{G}(w)$ Is the Set of Admissible Strategies (c, π, τ) in the COSOCP and Wealth Evolution is Given by

$$dW(t) = W(t)\left[r + \pi(t)(\mu - r)\right] dt + W(t)\pi(t)\sigma dB(t) - c(t) dt$$

Total Expected Discounted Utility $J_{c,\pi,\tau}(w)$:

$$E\left[\int_0^{T_x} e^{-\delta^S t}\left\{U_1(c(t))\,1_{\{t\le\tau\}} + U_2\left(\frac{W(\tau)}{\bar{a}_{x+\tau}}\right)1_{\{t>\tau\}}\right\}dt\right.$$

$$\left.+\eta e^{-\delta^S T_x}\left\{U_3\left(W(T_x)+Z^s\right)1_{\{T_x\le\tau\}} + U_3(Z^s)\,1_{\{T_x>\tau\}}\right\}\right]$$

with $Z^s = Z^{prior} - \underline{Z}$

COSOCP

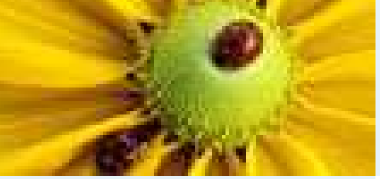
General COSOCP with Exponential Mortality:

$$V(w) = \sup_{(c,\pi,\tau)\in\mathcal{G}(w)} J_{c,\pi,\tau}(w) \qquad \text{for all } w > 0$$

$$J_{c,\pi,\tau}(w) = E^{w}\left[\int_0^{\tau} e^{-\beta^S t} f(c(t), W(t))\, dt + e^{-\beta^S \tau} g(W(\tau))\right]$$

with $\beta^S = \delta^S + \lambda^S$ and

$$dW(t) = W(t)\left[r + \pi(t)(\mu - r)\right] dt - c(t)\, dt + \sigma\pi(t) W(t)\, dB(t)$$



Verification Theorem - Optimal Strategies

- Annuitisation rule

$$\tau^* = \inf \{t \geq 0 | W^* (t) \notin D\}$$

with the Continuation Region

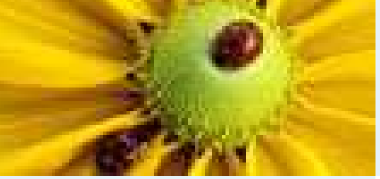
$$D = \{W(t) \in G \,|\, v(W(t)) > g(W(t))\}$$

- Consumption rule

$$c^* = I\left(v_W\left(W^*(t)\right)\right) 1_{\{t \leq \tau^*\}} \text{ with } I(.) = \left(\frac{\partial f}{\partial c}(.)\right)^{-1}$$

- Investment rule

$$\pi^* = -\frac{\mu - r}{\sigma^2} \frac{v_W\left(W^*(t)\right)}{W^*(t)\, v_{WW}\left(W^*(t)\right)} 1_{\{t \leq \tau^*\}}$$



COSOCP - Variational Inequality

The Verification Theorem Reduces the COSOCP to the Variational Inequality:

$$\max\left\{L^{\text{com}}v\left(W\left(t\right)\right), g\left(W\left(t\right)\right)-v\left(W\left(t\right)\right)\right\}=0 \quad \text{for } W\left(t\right)>0$$

with

$$L^{\text{com}}v\left(W\left(t\right)\right)=\sup_{(c,\pi)\in\mathcal{G}^{\tau}(W(t))}\left\{f\left(c\left(t\right),W\left(t\right)\right)-\beta^{S}v\left(W\left(t\right)\right)+Lv\left(W\left(t\right)\right)\right\}$$

subject to

$$v\left(W\left(t\right)\right)=g\left(W\left(t\right)\right) \quad \text{for all } W\left(t\right)\in\partial D$$

and

$$v_{W}\left(W\left(t\right)\right)=g_{W}\left(W\left(t\right)\right) \quad \text{for all } W\left(t\right)\in\partial D.$$

Continuation Region - Properties

- Continuation Region D Is Open and Connected
- $U \subset D$ with

$$U = \left\{ W(t) \in \mathbb{R}^{+} \mid L^{\text{com}} g(W(t)) > 0 \right\}$$

- The Set U Can Be Used to Infer Information about the Form of the Important Continuation Region D.
  - If $U \subsetneq D$: Continue When Wealth Falls out of U.
  - If $U = D$: Annuitise Immediately When Wealth Exits U.

  $\longrightarrow$Even in the Former Case it is Often Possible to Infer Important Information about the Form of the Crucial Continuation Region D by Studying the Set U.

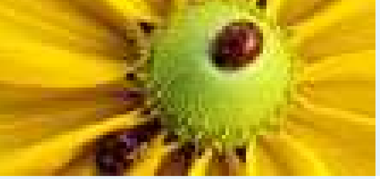
Indirect Utility - Exponential Mortality

Indirect Utility Function with Exponential Mortality:

$$E^{w}\left[\int_{0}^{\tau} e^{-\beta_x^S t}\left\{U_1\left(c\left(t\right)\right)+\lambda_x^S \eta U_3\left(W\left(t\right)+Z^s\right)\right\}dt\right.$$
$$\left.+e^{-\beta_x^S \tau}\frac{1}{\beta_x^S}\left\{U_2\left(W\left(\tau\right)\left(r+\lambda_x^O\right)\right)+\lambda_x^S \eta U_3\left(Z^s\right)\right\}\right]$$

$\rightarrow$ Calculate $L^{\text{com}} g\left(W\left(t\right)\right)$ to Determine the Set U

$\rightarrow$ Infer Information from U on the Continuation Region D



Derivation of the Set U - Power Utility

Assuming Power Utility Functions We Obtain

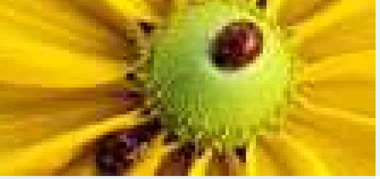
$$L^{\text{com}} g\left(W\left(t\right)\right) = \frac{W\left(t\right)^{1-\gamma}}{1-\gamma}\left[\gamma K_0^{-\frac{1-\gamma}{\gamma}} - \gamma K_0 K_2^{-1}\right] + \frac{\lambda^S \eta}{1-\gamma}\left[\left(W\left(t\right)+Z^s\right)^{1-\gamma} - \left(Z^s\right)^{1-\gamma}\right]$$

with the Constants

$$K_0 = \frac{\left(r+\lambda^O\right)^{1-\gamma}}{\beta^S} > 0,$$

$$K_2 = \frac{\gamma}{\beta^S - \left(1-\gamma\right)\left(r+\frac{\kappa}{\gamma}\right)}$$

$$\kappa = \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 > 0.$$

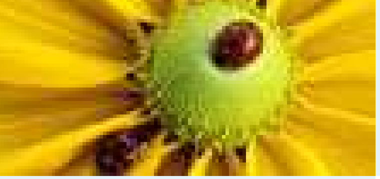


No-Bequest Case: General Results

- $U = D = \emptyset$ or $U = D = \mathbb{R}^+$ Dependent on M^{nb}
  $\longrightarrow$ Now-or-Never Annuitisation
  $\longrightarrow$ Trivial or Pure Optimal Control Problem
- M^{nb} Depends on Seven Parameters in a Non-Linear Way

$$M^{nb} = \gamma\left(\lambda^S + \delta^S\right)^{\frac{1-\gamma}{\gamma}}\left(r + \lambda^O\right)^{-\frac{(1-\gamma)^2}{\gamma}} - \left(r + \lambda^O\right)^{1-\gamma} + \left(r + \frac{1}{\gamma}\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)\frac{\left(r + \lambda^O\right)^{1-\gamma}}{\lambda^S + \delta^S}(1-\gamma)\,.$$

- Natural Parameter Effects
  - Risk Aversion: A$+$
  - Subjective Life Expectancy: A$+$
  - Objective Life Expectancy: A$-$
  - Identical Life Expectancy: A$-$
  - Sharpe Ratio: A$-$ (Only Unambiguous Effect)



No-Bequest Case: Numerical Example

Minimum Sharpe Ratio for Staying in the Financial Market for Identical Life Expectancy and Risk Aversion Combinations with $\delta^S = r = 0.035$

	$\gamma = 0.7$	$\gamma = 0.9$	$\gamma = 1.2$	$\gamma = 1.6$	$\gamma = 2$	$\gamma = 4$
$E[T] = 5$	0.5290	0.6000	0.6928	0.8000	0.8944	1.2649
$E[T] = 10$	0.3746	0.4242	0.4898	0.5656	0.6324	0.8944
$E[T] = 15$	0.3055	0.3464	0.4000	0.4618	0.5163	0.7302
$E[T] = 20$	0.2645	0.3000	0.3464	0.4000	0.4472	0.6324
$E[T] = 25$	0.2366	0.2683	0.3098	0.3577	0.4000	0.5656
$E[T] = 30$	**0.2160**	0.2449	0.2828	0.3265	0.3651	0.5163

$\longrightarrow$ Risk Aversion: A$+$

$\longrightarrow$ Identical Life Expectancy: A$-$

Assuming $\mu = 0.08$ and $\sigma = 0.2$ Implies Sharpe Ratio of 0.225.

$\Rightarrow$ Annuitisation Is Chosen in Most Parameter Settings

Bequest Case $\gamma < 1$, $Z^s = 0$: General Results

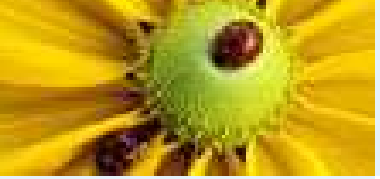
- Now-or-Never Annuitisation: $M^b = M^{nb} + \lambda^S \eta$
- Slight Tendency for the Financial Market
  $\rightarrow$ Important Inclusion of Bequest Motive (A$-$)
- Natural Parameter Effects
- Natural Comparison to No-Bequest Case
  - $\frac{c^b}{W^b} < \frac{c^{nb}}{W^{nb}}$
  - $W^b > W^{nb}$
  - $\pi^b = \pi^{nb}$
  - $\frac{c^b}{W^b}$ decreases in η

Bequest Case $\gamma < 1$, $Z^s = 0$: Numerical Ex. (1)

Minimum Sharpe Ratio for Staying in the Financial Market for Different Combinations of the Identical Life Expectancy and the Bequest Motive for $\gamma = 0.8$

	$\eta = 0$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
$E[T] = 5$	0.5656	0.4409	0.2623	**0.0000**	**0.0000**
$E[T] = 10$	0.4000	0.3459	0.2817	**0.1977**	**0.0000**
$E[T] = 15$	0.3265	0.2919	0.2526	**0.2059**	**0.1449**
$E[T] = 20$	0.2828	0.2570	0.2284	**0.1955**	**0.1559**
$E[T] = 25$	0.2529	0.2322	**0.2094**	**0.1838**	**0.1539**
$E[T] = 30$	0.2309	**0.2134**	**0.1942**	**0.1730**	**0.1488**
$E[T] = 35$	**0.2138**	**0.1985**	**0.1819**	**0.1637**	**0.1431**

$\longrightarrow$ Bequest Motive: A–
$\longrightarrow$ Identical Life Expectancy: Normally A–
$\Longrightarrow$ Slight Tendency for the Financial Market



Bequest Case $\gamma < 1$, $Z^s = 0$: Numerical Ex. (2)

Mortality Rate Transformation: $\lambda^S = \lambda^O(1 + l)$
Minimum Markup Parameter l for Staying in the Financial Market for Objective Life Expectancy and Bequest Motive Combinations Assuming $\gamma = 0.8$

	$\eta = 0$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
$E^O[T] = 5$	1.4089	0.4191	0.0379	$\mathbf{-0.1760}$	$\mathbf{-0.3161}$
$E^O[T] = 10$	1.3395	0.5557	0.1760	$\mathbf{-0.0573}$	$\mathbf{-0.2179}$
$E^O[T] = 15$	1.2292	0.5359	0.1606	$\mathbf{-0.0826}$	$\mathbf{-0.2557}$
$E^O[T] = 20$	1.0681	0.4167	0.0337	$\mathbf{-0.2308}$	$\mathbf{-0.4348}$
$E^O[T] = 25$	0.8316	0.1666	$\mathbf{-0.3529}$	$\mathbf{-0.5449}$	$\mathbf{-0.6189}$
$E^O[T] = 30$	0.4180	$\mathbf{-0.2325}$	$\mathbf{-0.3708}$	$\mathbf{-0.4780}$	$\mathbf{-0.5646}$
$E^O[T] = 35$	0.0855	$\mathbf{-0.1321}$	$\mathbf{-0.2901}$	$\mathbf{-0.4126}$	$\mathbf{-0.5117}$

$\longrightarrow$ Bequest Motive: $A-$
$\longrightarrow$ Markup Parameter l: $A-$
$\Rightarrow$ Pensioner Rejects Even Some Favourable Annuities

Bequest Case $\gamma > 1$, $Z^s > 0$: General Results

- Never Annuitisation or Wealth-Dependent Annuitisation with $D = (\underline{W}, \infty)$
- Natural Comparison to No-Bequest Case
- Real COSOCP with $D = (\underline{W}, \infty)$:
  - More Involved Problem
    - Simplification via Duality Arguments
    - Free Boundary Value Problem
    - Numerical Solution Algorithm
      $\longrightarrow$ Boundaries
      $\longrightarrow$ Value Function
  - Natural Parameter Effects:
    $\longrightarrow$ Life Insurance: $A+$
    $\longrightarrow$ Bequest Motive: $A-$
  - Heavy Consumption Smoothing
  - More Aggressive Investment Rule Compared to Merton
    $\longrightarrow$ Additional Option of Annuitisation

Bequest Case $\gamma > 1$, $Z^s > 0$: Problem

- The Problem Involves Solving $L^{\text{com}} u(W(t)) = 0$

$$
\begin{aligned}
0 &= \frac{\gamma}{1-\gamma}\left[u_W(W(t))\right]^{-\frac{1-\gamma}{\gamma}} + \lambda^S \eta \frac{\left(W(t)+Z^s\right)^{1-\gamma}}{1-\gamma} \\
&\quad -\beta^S u(W(t)) + rW(t)\, u_W(W(t)) - \kappa \frac{\left[u_W(W(t))\right]^2}{u_{WW}(W(t))}.
\end{aligned}
$$

$\rightarrow$ Highly Non-Linear ODE for u

- Using Duality Arguments We Obtain

$$
\begin{aligned}
0 &= \frac{\gamma}{1-\gamma} y(t)^{-\frac{1-\gamma}{\gamma}} + \lambda^S \eta \frac{\left[-\tilde{u}_y(y(t)) + Z^s\right]^{1-\gamma}}{1-\gamma} \\
&\quad -\beta^S \tilde{u}(y(t)) + \tilde{u}_y(y(t))\, y(t)\left(\beta^S - r\right) + \kappa \tilde{u}_{yy}(y(t))\, y(t)^2 .
\end{aligned}
$$

$\rightarrow$ Only Slightly Non-Linear ODE for $\tilde{u}$

Bequest Case $\gamma > 1$, $Z^s > 0$: Solution Algorithm

- No Analytical Solution to the ODE for $\tilde{u}$ on the Dual Continuation Region $\tilde{D} = \left(0, \underline{y}\right)$.
  $\longrightarrow$ Numerical Solution Algorithm
  $\longrightarrow$ Need Explicit Boundary Value Conditions
  $\longrightarrow$ Use Smooth Paste and Smooth Fit Condition
- Problem: Thresholds $\underline{y}$ and $\underline{W}$ Unknown (FBVP)
  $\longrightarrow$ Algorithm Must Simultaneously Solve for the Dual Function $\tilde{u}$ and the Boundaries $\underline{y}$ and $\underline{W}$
  $\longrightarrow$ Central Idea: Probability of Annuitisation (Wealth Falling Below $\underline{W}$) Vanishes as $W \to \infty$
  $\longrightarrow$ Constant Merton Investment Rule in the Limit
  $\longrightarrow$ Exploiting this Condition We Can Construct an Iterative Algorithm on the Bisection Method

Bequest Case $\gamma > 1$, $Z^s > 0$: Investment

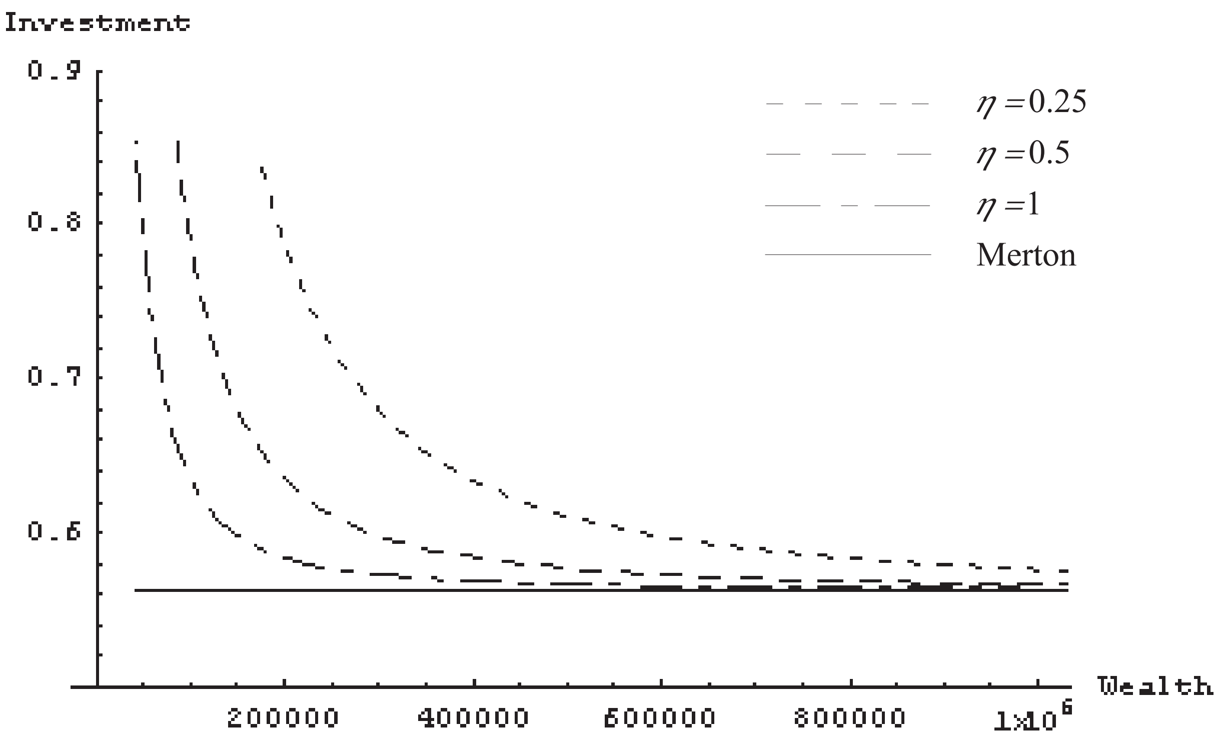


Figure 1: Investment Rule for Different Bequest Parameters Assuming a Subjective and Objective Life Expectancy of 20 Years, $\delta^S = r = 0.035$, $\mu = 0.08$, $\sigma = 0.2$, Life Insurance Net of Subsistence of 500 and $\gamma = 2$.

Bequest Case $\gamma > 1$, $Z^s > 0$: Consumption

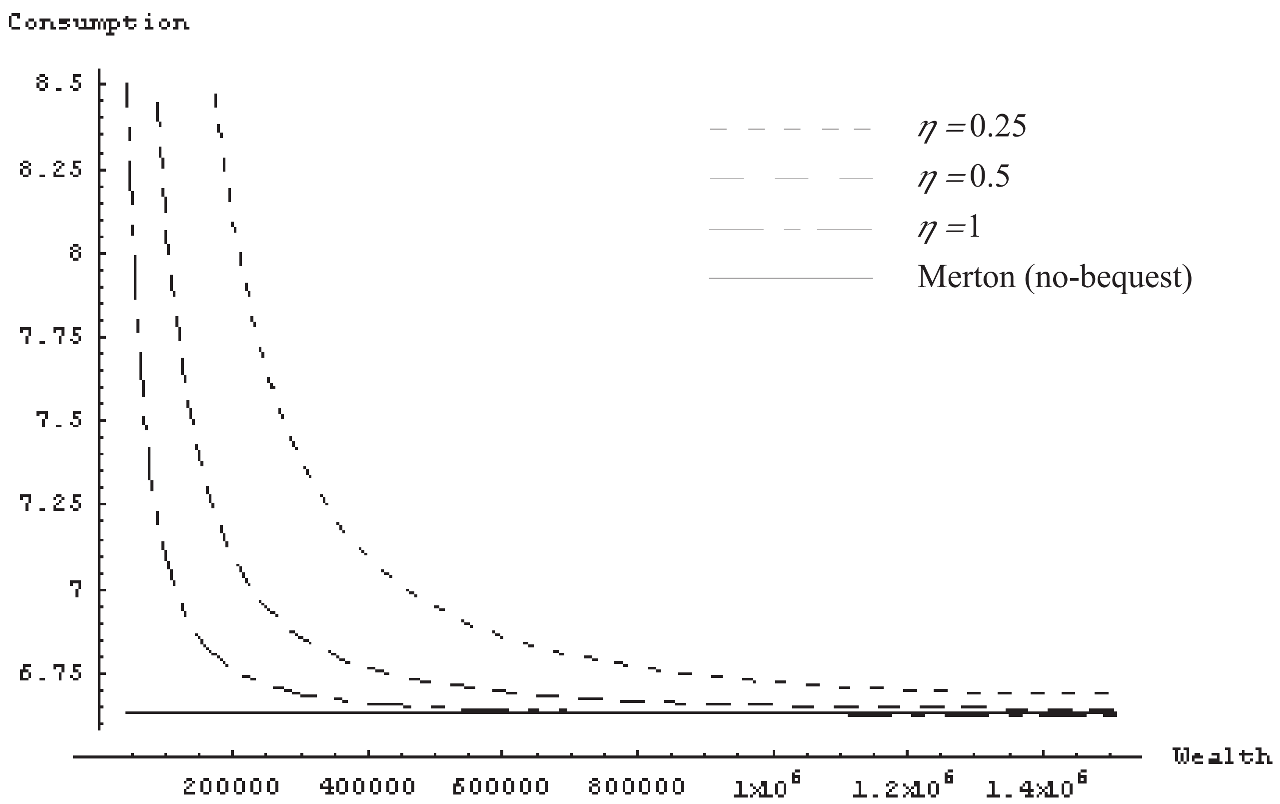
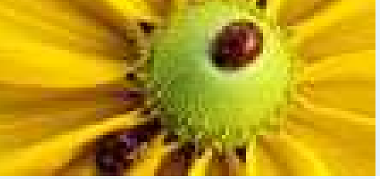
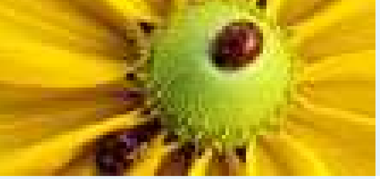


Figure 2: Consumption Fraction for Different Bequest Parameters Assuming a Subjective and Objective Life Expectancy of 20 Years, $\delta^S = r = 0.035$, $\mu = 0.08$, $\sigma = 0.2$, Life Insurance Net of Subsistence of 500 and $\gamma = 2$.



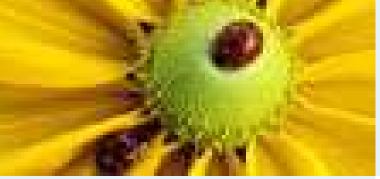
Main Conclusions

- COSOCP: New Solution Method
- Economically Important Risk Aversion $\gamma > 1$
- Longevity Risk Is Absolutely Relevant
  $\longrightarrow$ Modelling of Lifetime
  $\longrightarrow$ Role of Pension Funds
- Essential Inclusion of a Bequest Motive
  $\longrightarrow$ Consumption-Wealth Trade-off
  $\longrightarrow$ Absurd Strong Tendency for the Annuity Market Vanishes



Thanks

Thank You Very Much for Your Attention!



Duality Arguments (1)

- Definition of the Convex Dual of $u\left(W\left(t\right)\right)$

$$\tilde{u}\left(y\left(t\right)\right) = \max_{W(t)>0}\left[u\left(W\left(t\right)\right) - y\left(t\right)W\left(t\right)\right]$$

$\rightarrow \; u_W\left(W^*\left(t\right)\right) = y\left(t\right)$
$\rightarrow \; W^*\left(t\right) = I\left(y\left(t\right)\right)$ with I: Inverse Function of $u_W\left(W\left(t\right)\right)$

- Implied Expression for the Convex Dual

$$\tilde{u}\left(y\left(t\right)\right) = u\left(W^*\left(t\right)\right) - y\left(t\right)W^*\left(t\right) = u\left(I\left(y\left(t\right)\right)\right) - y\left(t\right)I\left(y\left(t\right)\right).$$

- First Order Derivative

$$\begin{aligned}\tilde{u}_y\left(y\left(t\right)\right) &= u_W\left(I\left(y\left(t\right)\right)\right)I_y\left(y\left(t\right)\right) - I\left(y\left(t\right)\right) - y\left(t\right)I_y\left(y\left(t\right)\right)\\ &= y\left(t\right)I_y\left(y\left(t\right)\right) - I\left(y\left(t\right)\right) - y\left(t\right)I_y\left(y\left(t\right)\right)\\ &= -I\left(y\left(t\right)\right) < 0\end{aligned}$$

Duality Arguments (2)

- It Follows that

$$W^*(t) = -\tilde{u}_y(y(t))$$

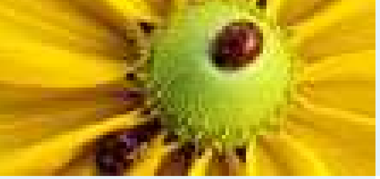
- Second Order Derivative

$$\tilde{u}_{yy}(y(t)) = -I_y(y(t)) = -\frac{1}{u_{WW}(I(y(t)))} > 0$$

$\rightarrow$ Strict Convexity of the Dual Function $\tilde{u}$ Follows From the Strict Concavity of the Primal Function u.

- Lastly, We Have

$$u(W^*(t)) = \tilde{u}(y(t)) + y(t)\,W^*(t) = \tilde{u}(y(t)) - \tilde{u}_y(y(t))\,y(t)$$



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