Shortfall Minimizing Portfolios[0]

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Abstract

Many institutional and private investors seek for a long run excess return relative to a reference strategy (e.g. money market, bond index, etc.) which they want to attain under a minimal shortfall probability. In this article it is shown that even in the long run in order to attain a substantial excess return a high shortfall probability has to be accepted.

In the model the prices of the assets follow geometric Brownian motions. Two types of a shortfall are distinguished. A shortfall of type I occurs, if at some point of time the investment goal is missed by a given percentage. There is a shortfall of type II, if the investment goal is missed at the end of the planning horizon. To begin with, only constant portfolio weights are admitted. For both types it can be shown that minimizing the shortfall probability under a given excess return is equivalent to the Merton problem. Under realistic parameter values moderate shortfall probabilities are only compatible with very low excess returns. Finally, it is shown, that "Constant Proportion Portfolio Insurance" does not lead to a reduction of the shortfall probability.

1 Introduction

Many private and institutional investors are interested in investment strategies which outperform some given reference rate of return. This reference rate may be deterministic (e.g. 4%) or stochastic (e.g. 1% plus rate of inflation). In case of a stochastic reference rate we shall assume that it can be tracked by an asset portfolio. However, due to the absence of arbitrage neither a riskless investment nor a tracking portfolio can be outperformed with probability one. Hence, if the reference rate is chosen above the corresponding level, then under any investment strategy a shortfall may occur. Therefore, a natural objective of an investor is to minimize the probability of a shortfall. In our paper, which is based on geometric Brownian motions for asset prices, we shall discuss two types of shortfall. A shortfall of the first type occurs if there is some point of time, where the investment strategy does not attain the reference level. A second type of shortfall occurs if at a fixed time horizon the return of the investment strategy is below the reference level.

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In order to avoid trivial solutions for the shortfall optimization of the second type one has to exclude e.g. strategies whose return is either the reference rate or a complete loss [1]. Therefore we look first of all at investment strategies with constant portfolio weights and it turns out that minimizing the shortfall probability for both types leads to solutions of the Merton problem for investors with constant relative risk aversion (Merton (1971)). As expected, for the second type of shortfall the optimal investment policy is less conservative than for the first type. However, for both types either the shortfall probability is high or the optimal investment policy is very conservative. Due to this result we want to admit a class of investment strategies with non constant portfolio weights. A natural candidate is constant proportion portfolio insurance (CPPI). It is well known that in the absence of borrowing limits and transaction costs CPPI can be represented as a solution of the Merton problem for an investor with a HARA utility function (Black and Perold (1992)). Using this result we show that optimal shortfall probabilities are worse than in the case of constant portfolio weights.

These results and calculations with realistic parameter values illustrate the difficulties to outperform substantially the riskless investment or a tracking portfolio. Either one has to accept a rather high shortfall risk or one has to use option strategies which may lead to heavy losses.

2 Model

There is a riskless investment opportunity $i = 0$ and risky investment opportunities $i = 1, ..., N$ whose price processes are given by geometric Brownian motions, i.e.

$$\begin{aligned} \frac{dS_{0,t}}{S_{0,t}} &= rdt \quad , \\ \left(\frac{dS_{i,t}}{S_{i,t}}\right)_{i=1,\dots,N} &= (r+\pi_i)_{i=1,\dots,N}\ dt + \sigma d\mathbf{Z}_t \quad , \end{aligned} \tag{1}$$

where $\mathbf{Z}_t$ denotes the N-dimensional standard Brownian motion,

σ is a regular matrix.

At each point of time t the wealth W_t is assumed to be invested in fixed proportions. Choosing a portfolio $\mathbf{x} \in \mathbb{R}^N$ therefore means that the amount $x_i W_t$ is invested in asset i, $(i = 1, ..., N)$, and $\left(1 - \sum_{i=1}^N x_i\right) W_t$ in the riskless asset. Then, under a portfolio choice $\mathbf{x} \in \mathbb{R}^N$ the wealth process W_t is given by

$$\frac{dW_t}{W_t} = \left(r + \boldsymbol{\pi}^T \mathbf{x}\right) dt + \mathbf{x}^T \sigma d\mathbf{Z}_t. \tag{2}$$

The wealth target L_t of the investor is given by

$$\frac{dL_t}{L_t} = \left(r + c + \boldsymbol{\pi}^T \mathbf{b}\right) dt + \mathbf{b}^T \sigma d\mathbf{Z}_t \quad , \ L_0 = W_0. \tag{3}$$

[1] Such a payoff can be achieved in a complete market by using appropriate option strategies.

Hence, the reference rate of return is given by the rate of return of a portfolio **b** (e.g. a bond index) plus a rate of outperformance c. For $\mathbf{b} = 0$ the reference rate is deterministic.

In order to define a shortfall we need the ratio

$$F_t = \frac{W_t}{L_t}.$$

After some calculations (see Appendix I) one gets

$$\frac{dF_t}{F_t} = \left[-c + \mathbf{y}^T \left(\boldsymbol{\pi} - \sigma\sigma^T \mathbf{b}\right)\right] dt + \mathbf{y}^T \sigma d\mathbf{Z}_t \ , \tag{4}$$

$$\text{with} \ \ \mathbf{y} = \mathbf{x} - \mathbf{b}$$

or

$$d \ln F_t = \left[-c + \mathbf{y}^T \left(\boldsymbol{\pi} - \sigma\sigma^T \mathbf{b}\right) - \frac{1}{2}\mathbf{y}^T \sigma\sigma^T \mathbf{y}\right] dt + \mathbf{y}^T \sigma d\mathbf{Z}_t \ , \tag{5}$$

$$F_0 = 1.$$

We assume that the investor chooses a critical level $a \in (0, 1]$ for F_t (e.g. $a = 0.90$) and define the two types of shortfall.

Shortfall of Type I

F_t attains a value below a for some $t > 0$, i.e.

$$\inf_{t>0} F_t < a.$$

Shortfall of Type II

At a fixed time horizon T,

$$F_T < a$$

holds.

From (5) one obtains

$$\ln F_t = g(\mathbf{y}) \cdot t + h(\mathbf{y}) \cdot \hat{Z}_t \ ,$$

with

$g(\mathbf{y}) = -c + \mathbf{y}^T(\boldsymbol{\pi} - \sigma\sigma^T \mathbf{b}) - \frac{1}{2}\mathbf{y}^T \sigma\sigma^T \mathbf{y}$,

$h(\mathbf{y}) = \left(\mathbf{y}^T \sigma\sigma^T \mathbf{y}\right)^{0.5}$,

$\hat{Z}_t$ univariate standard Brownian motion.

Obviously, the shortfall criteria can be written in the form

$$\text{(SI)} \quad \inf_{t>0}(\ln F_t) < \ln a \quad \text{for } a \text{ shortfall of type I} \ , \tag{6}$$

$$\text{(SII)} \quad \ln F_T < \ln a \quad \text{for } a \text{ shortfall of type II} \ . \tag{7}$$

3 Shortfall Optimization

3.1 Minimizing the Shortfall Probability of Type I

For $g(\mathbf{y}) \leqslant 0$ the probability of a shortfall of type I is one. For $g(\mathbf{y}) > 0$ the shortfall probability can be derived from Gerber (1975), p. 316, example 1 or from Baumann (2005), pp. 49-50.

$$\text{Prob}\left[\inf_{t>0}(\ln F_t) < \ln a\right] = a^{\frac{2g(\mathbf{y})}{[h(\mathbf{y})]^2}} \;\;,\;\; 0 < a < 1 \tag{8}$$

for $g(\mathbf{y}) > 0$.

Therefore, in order to minimize the shortfall probability one has to solve

$$\max_{\mathbf{y}\in\mathbb{R}^N} \frac{g(\mathbf{y})}{[h(\mathbf{y})]^2}. \tag{9}$$

Proposition 1 *For*

$$0 < c < \frac{1}{2}\sigma^2_{\mathbf{x}^*,\mathbf{b}} \;\;,\;\; 0 < a < 1\;, \tag{10}$$

$$\begin{aligned} \textit{with}\;\; \sigma^2_{\mathbf{x}^*,\mathbf{b}} &= (\mathbf{x}^* - \mathbf{b})^T\, V\, (\mathbf{x}^* - \mathbf{b})\,, \\ V &= \sigma\sigma^T \;\;,\; \mathbf{x}^* = V^{-1}\boldsymbol{\pi} \end{aligned}$$

the minimal shortfall probability is smaller than one and is attained at

$$\mathbf{x}^I = \mathbf{b} + \frac{2c}{\sigma^2_{\mathbf{x}^*,\mathbf{b}}} \cdot (\mathbf{x}^* - \mathbf{b})\,. \tag{11}$$

Proof. See Appendix I. ■

Comments

1. Here $\mathbf{x}^* = V^{-1}\boldsymbol{\pi}$ denotes the growth optimum portfolio which results from maximizing $E\,[\ln W_t]$ and which has the following properties (see e.g. Panjer, H. et al. 2001, p.141-143):
   - i) $\mathbf{x}^*$ maximizes the median of W_t for all t.
   - ii) $\boldsymbol{\pi}^T\mathbf{x}^* = \mathbf{x}^{*T}V\mathbf{x}^* = \boldsymbol{\pi}^T V^{-1}\boldsymbol{\pi}$.
2. Special case: $\mathbf{b} = \mathbf{0}$

   In this case the investor wants to beat the riskless investment by c. Due to (10)

   $$c < \frac{1}{2}\pi_{\mathbf{x}^*} = \frac{1}{2}\sigma^2_{\mathbf{x}^*}$$

   must hold, where $\pi_{\mathbf{x}^*}$ denotes the risk premium and $\sigma_{\mathbf{x}^*}$ the volatility of the growth optimum portfolio. According to (11) the portfolio $\mathbf{x}^I$ is the Merton solution for an investor with a constant relative risk aversion of

   $$\gamma^I = \frac{\pi_{\mathbf{x}^*}}{2c}.$$

A straightforward calculation (Appendix I) leads to

$$\frac{g(\mathbf{y}^I)}{[h(\mathbf{y}^I)]^2} = \frac{\pi_{\mathbf{x}^*}}{4c} - \frac{1}{2}. \tag{12}$$

Therefore the shortfall probability is given by

$$\text{Prob}\left[\inf_{t>0}(\ln F_t) < \ln a\right] = a^{\frac{\pi_{\mathbf{x}^*}}{2c}-1} .$$

3. General case: $\mathbf{b} \neq \mathbf{0}$

   If $\mathbf{b} \neq \mathbf{x}^*$ and if the rate of outperformance satisfies

   $$c < \frac{1}{2}\sigma^2_{\mathbf{x}^*,\mathbf{b}} ,$$

   then a shortfall of type I occurs with probability (see Appendix I)

   $$a^{\frac{\sigma^2_{\mathbf{x}^*,\mathbf{b}}}{2c}-1} . \tag{13}$$

   For the case where $\mathbf{b}$ tracks a price index, Adler and Dumas (1983) show that (11) is the optimal portfolio for an investor who optimizes real wealth with a constant relative risk aversion

   $$\gamma^I = \frac{\sigma^2_{\mathbf{x}^*,\mathbf{b}}}{2c} .$$

Numerical Example

$\mathbf{b} = \mathbf{0}$, $\pi_{\mathbf{x}^*} = 6.25\%$, $(\sigma_{\mathbf{x}^*} = 25\%)$

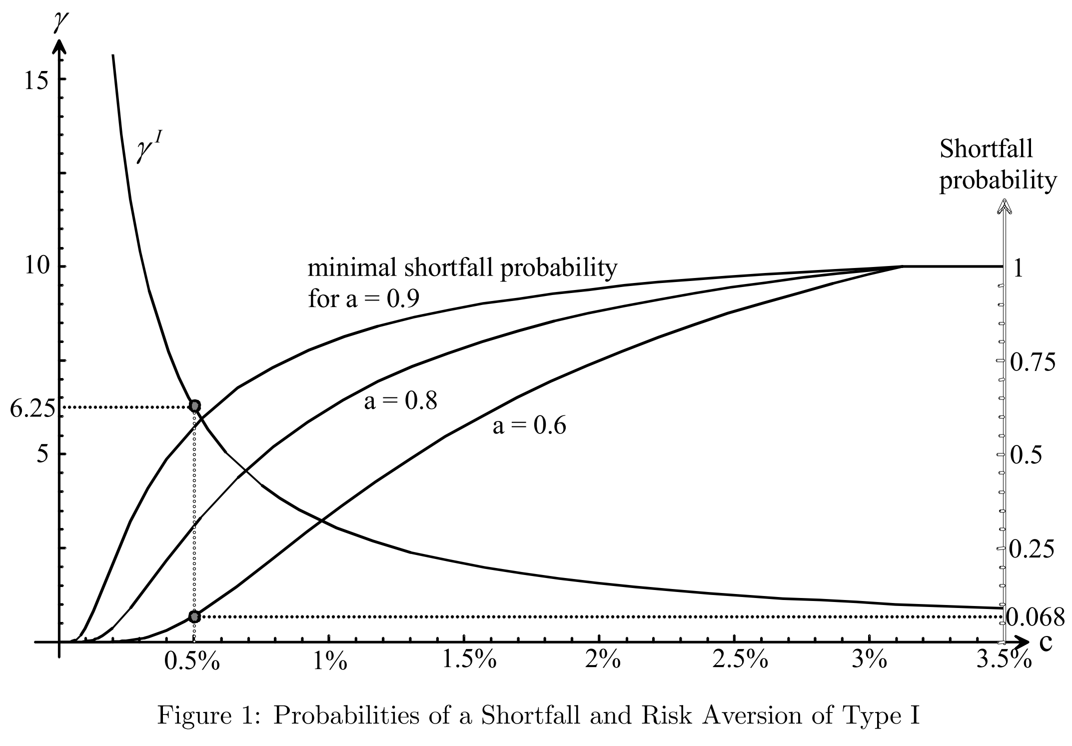

Figure 1: Probabilities of a Shortfall and Risk Aversion of Type I

The probabilities of a shortfall for this example are shown in Figure 1. For the growth optimum portfolio $\mathbf{x}^*$ a risk premium $\pi_{\mathbf{x}^*} = 6.25\%$ which corresponds to a volatility $\sigma_{\mathbf{x}^*} = 25\%$ seems to be realistic. As Figure 1 shows, high probabilities of a shortfall can only be avoided, if one accepts rather low rates of outperformance. An exception is perhaps the case $a = 0.6$, $c = 0.5\%$ which corresponds to a shortfall probability of 0.0684 and a relative risk aversion of 6.25 and therefore to a rather conservative investment strategy.

3.2 Minimizing the Shortfall Probability of Type II

If we concentrate only on a shortfall at time horizon T one expects lower shortfall probabilities and a less conservative investment strategy. A shortfall of type II occurs if

$$\ln F_T < \ln a \quad , \quad 0 < a \leqslant 1$$

or

$$g(\mathbf{y}) \cdot T + h(\mathbf{y})\hat{Z}_T < \ln a \ .$$

The shortfall probability is

$$\begin{aligned} \Phi(z(\mathbf{y})) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(\mathbf{y})} e^{-\frac{x^2}{2}} dx \quad , \\ \text{with } z(\mathbf{y}) &= \frac{1}{\sqrt{T} \cdot h(\mathbf{y})} \cdot (\ln a - T g(\mathbf{y})) \ . \end{aligned}$$

In order to minimize the probability of a shortfall of type II one has to solve

$$\min_{\mathbf{y} \in \mathbb{R}^N} \frac{\sqrt{T}}{h(\mathbf{y})} \cdot \left(\frac{\ln a}{T} - g(\mathbf{y}) \right) \ . \tag{14}$$

Proposition 2 *Assume*

$$0 < c + \frac{\ln a}{T} < \frac{1}{2}\sigma^2_{\mathbf{x}^*,\mathbf{b}} \ . \tag{15}$$

Then the minimal shortfall probability is smaller than 0.5 *and is attained at*

$$\mathbf{x}^{II} = \mathbf{b} + \frac{1}{\sigma_{\mathbf{x}^*,\mathbf{b}}} \sqrt{2\left(c + \frac{\ln a}{T}\right)} \cdot (\mathbf{x}^* - \mathbf{b}) \ . \tag{16}$$

Proof. See Appendix II. ■

Comments

1. For $c + \frac{\ln a}{T} \leqslant 0$ one could invest in the portfolio $\mathbf{b}$ or in the riskless asset and avoid with probability one a shortfall of type II. To see this, observe that putting $\mathbf{y} = \mathbf{0}$ in (5) leads to $F_T = F_0 e^{-cT}$. However, $c + \frac{\ln a}{T} \leqslant 0$ implies $e^{-cT} \geqslant a$.

2. Special case $\mathbf{b} = \mathbf{0}$:

   Again the Merton solution results. For $a = 1$ the implied relative risk aversion is

   $$\gamma^{II} = \left(\frac{\pi_{\mathbf{x}^*}}{2c}\right)^{0.5} .$$

   In comparison with section 3.1 we get $\gamma^{II} < \gamma^I$ for $0 < c < \frac{\pi_{\mathbf{x}^*}}{2}$. As expected, shortfall optimization of type II leads to less conservative investment strategies.

3. In the general case the shortfall probability (see Appendix II) is

   $$\Phi\left[\sqrt{T}\left(\sqrt{2\left(\frac{\ln a}{T} + c\right)} - \sigma_{\mathbf{x}^*,\mathbf{b}}\right)\right] \tag{17}$$

   and x^{II} is the Merton solution for an investor optimizing F_T with a constant relative risk aversion

   $$\gamma^{II} = \frac{\sigma_{\mathbf{x}^*,\mathbf{b}}}{\sqrt{2\left(c + \frac{\ln a}{T}\right)}} .$$

In the following example, the shortfall probabilities are again calculated for realistic parameter values.

Numerical Example

$\mathbf{b} = \mathbf{0}$, $a = 1$, $T = 20$, $\pi_{\mathbf{x}^*} = 6.25\%$ ($\sigma_{\mathbf{x}^*} = 25\%$) .

c	2%	1%	0.5%	0.25%	0.10%
γ^{II}	1.250	1.768	2.500	3.536	5.590
z	−0.223	−0.485	−0.670	−0.802	−0.918
$\Phi(z)$	0.4117	0.3138	0.2514	0.2113	0.1793

Table 1: Probabilities of a Shortfall of Type II

According to table 1 even a low rate of excess performance (0.10%) leads to a rather high shortfall probability (17.93%). These results are also shown in Figure 2.

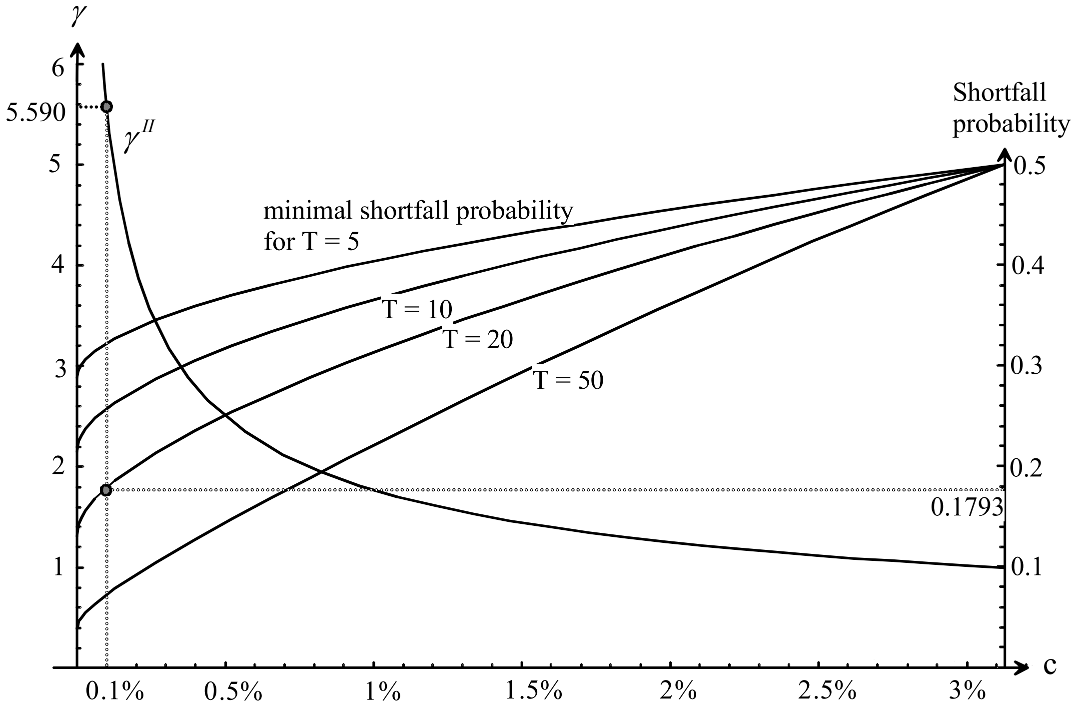

Figure 2: Probabilities of a Shortfall and Risk Aversion of Type II for $a = 1$

3.3 Maximizing the $\alpha-$Percentile at Time T

Maximization of percentiles contains value at risk minimization as a special case. In this section the $\alpha-$percentile of F_T is maximized ($\alpha \leqslant 0.5$). In section 2 we derived

$$\ln F_T = g(\mathbf{y}) \cdot T + h(\mathbf{y}) \cdot \hat{Z}_T \ .$$

Therefore maximizing the $\alpha-$percentile of F_T leads to the optimization problem

$$\max_{\mathbf{y} \in \mathbb{R}^N} \left[g(\mathbf{y}) \cdot T - |z_\alpha| \cdot \sqrt{T} \cdot h(\mathbf{y}) \right] \quad ,$$

$$\text{with} \quad \Phi(z_\alpha) = \alpha$$

or

$$\max_{\mathbf{y} \in \mathbb{R}^N} \left[-c + \mathbf{y}^T (\boldsymbol{\pi} - V\mathbf{b}) - \frac{1}{2} \mathbf{y}^T V \mathbf{y} - \frac{|z_\alpha|}{\sqrt{T}} \cdot \left(\mathbf{y}^T V \mathbf{y} \right)^{0.5} \right] \ . \tag{18}$$

Proposition 3 *The $\alpha-$percentile of F_T is maximized for*

$$\mathbf{x}^{III} = \begin{cases} \mathbf{b} & \text{if } \alpha \leqslant \Phi\left(-\sqrt{T}\sigma_{\mathbf{x}^*,\mathbf{b}}\right) \\ \mathbf{b} + \left(1 - \dfrac{|z_\alpha|}{\sqrt{T}\sigma_{\mathbf{x}^*,\mathbf{b}}}\right) \cdot (\mathbf{x}^* - \mathbf{b}) & \text{if } \alpha > \Phi\left(-\sqrt{T}\sigma_{\mathbf{x}^*,\mathbf{b}}\right) . \end{cases}$$

Proof. See Appendix III ■

Comments

1. For small values of α the optimal solution is to invest in the tracking portfolio or in the riskless asset.

2. Increases in α and T lead to a less conservative investment strategy. The investment strategy does not depend on the rate of outperformance c.

3. The objective function in (18) is concave. Therefore, even under linear constraints, the optimization problem can be easily solved numerically. If the optimization problem of section 3.2 has to be solved under linear constraints, one can apply percentile optimization for different levels of α . Using a bisection method with respect to α , the shortfall optimization problem can be solved easily.

In the next example optimal portfolios for different levels of α are calculated.

Numerical Example

$$\mathbf{b} = \mathbf{0} \ \ , \ \ a = 1 \ \ , \ \ T = 20 \ \ , \pi_{\mathbf{x}^*} = 6.25\% \ \ , c = 1\% \quad (\sigma_{\mathbf{x}^*} = 25\%) \ \ .$$

In Figure 3 it is shown that the investment strategy $\mathbf{x}^{III}$ resulting from the maximization of the α-percentile of F_T, $T = 20$ is highly sensitive to α. For $\alpha \leqslant 0.1318$ the riskless investment strategy results and F_t decreases with the rate $c = 1\%$. For $\alpha = 0.5$ the median of F_T, $T = 20$ is maximized and the growth optimum portfolio results, i.e. $\mathbf{x}^{III} = \mathbf{x}^*$.

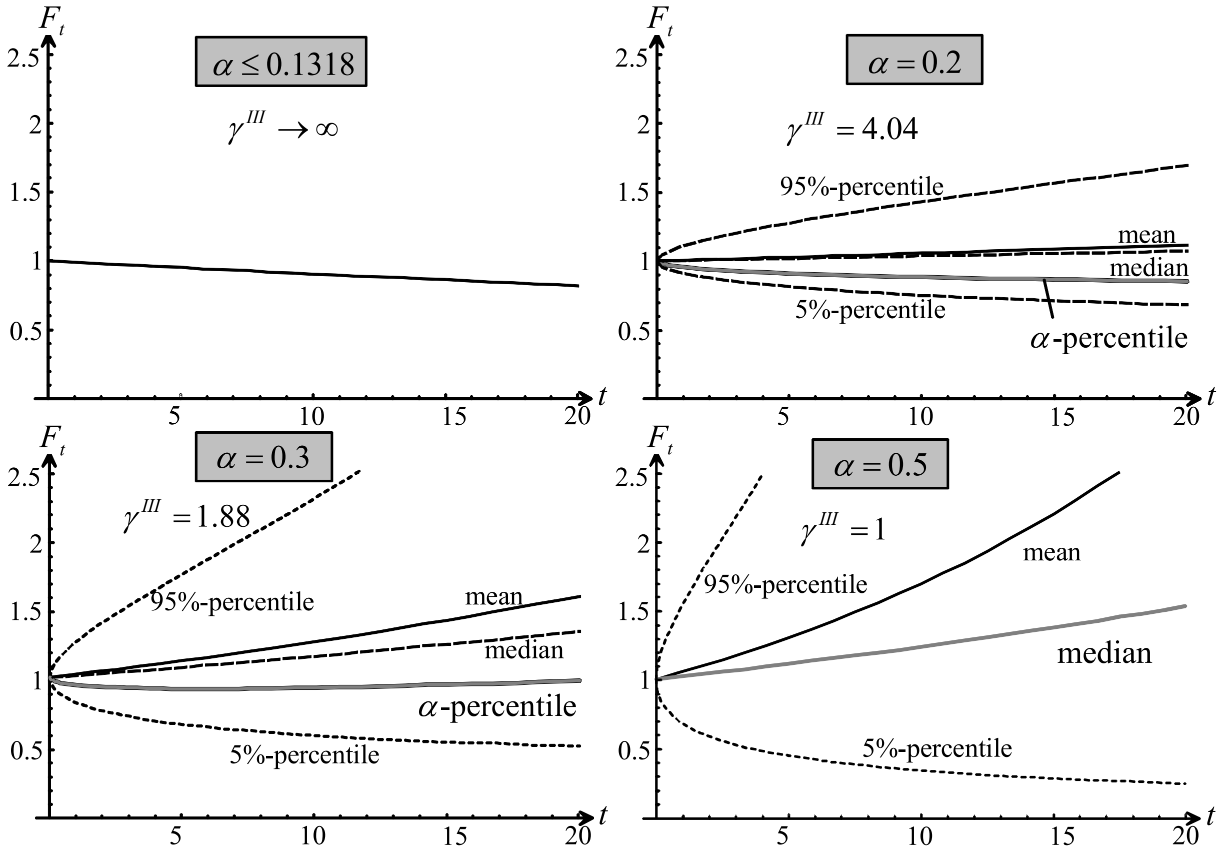

Figure 3: Percentile optimization for different values of α.

4 Downside Protection by Constant Proportion Portfolio Insurance

From section 3 it became clear that in case of constant portfolio weights shortfall minimization, respectively maximization of percentiles leads either to rather conservative strategies or to high shortfall probabilities. Therefore it may be of interest to look at a model of capital protection. Black, F. and Perold, A.F. (1992) proposed the method of Constant Proportion Portfolio Insurance (CPPI). In their model wealth W_t is split into a floor A_t and a cushion C_t . The price processes for the investment opportunities are still given by (1). But for the wealth target L_t we assume

$$\frac{dL_t}{L_t} = (r+c)dt \quad , \quad L_0 = W_0. \tag{19}$$

In CPPI the floor is invested in the riskless asset. Therefore

$$A_t = A_0 \cdot e^{rt}$$

holds.

Furthermore, we assume that there is no limit on borrowing and that the cushion is invested in the growth optimum portfolio $\mathbf{x}^* = V^{-1}\boldsymbol{\pi}$ leveraged by a factor m. If e.g., $\mathbf{x}^*$ consists of 120% equities, then for $m = 1.5$ one had to invest 180% in equities and to go short 80% in the riskless investment opportunity (money market).

The investment strategy $m\mathbf{x}^* = mV^{-1}\boldsymbol{\pi}$ leads to

$$\frac{dC_t}{C_t} = (r + m \cdot \pi_{\mathbf{x}^*})dt + m\sigma_{\mathbf{x}^*} d\hat{Z}_t \quad ,$$

$$\text{where} \quad \pi_{\mathbf{x}^*} = \sigma^2_{\mathbf{x}^*} = \boldsymbol{\pi}^T V^{-1} \boldsymbol{\pi}$$

or

$$d\ln C_t = (r + m \cdot \pi_{\mathbf{x}^*} - \frac{m^2}{2}\sigma^2_{\mathbf{x}^*})dt + m\sigma_{\mathbf{x}^*} d\hat{Z}_t.$$

Therefore one obtains

$$C_t = C_0 \cdot e^{\left[r + \left(m - \frac{m^2}{2}\right)\sigma^2_{\mathbf{x}^*}\right]t + m \cdot \sigma_{\mathbf{x}^*} \hat{Z}_t},$$

$$C_0 = W_0 - A_0.$$

Total wealth under CPPI is given by

$$W_t = A_0 \cdot e^{rt} + (W_0 - A_0)\, e^{\left[r + \left(m - \frac{m^2}{2}\right)\sigma^2_{\mathbf{x}^*}\right]t + m\sigma_{\mathbf{x}^*} \hat{Z}_t}.$$

It is well known that an investor with time horizon T and a utility function of the type

$$u(w) = \begin{cases} \dfrac{1}{1-\gamma}(w-k)^{1-\gamma} \quad , & \gamma > 0 \ \ , \ \ \gamma \neq 1 \\ \ln(w-k) \ , & \gamma = 1 \end{cases}$$

chooses a CPPI strategy with

$$A_0 = k \cdot e^{-rT},$$
$$m = \frac{1}{\gamma}.$$

For the analysis of a shortfall at T we have to deal with

$$F_T = \frac{W_T}{L_T} = \frac{A_0}{W_0} \cdot e^{-cT} + \left(1 - \frac{A_0}{W_0}\right) e^{\left[-c + \left(m - \frac{m^2}{2}\right)\sigma^2_{\mathbf{x}^*}\right]T + m\sigma_{\mathbf{x}^*} \hat{Z}_T}. \tag{20}$$

Proposition 4 *Assume*

$$0 < c + \frac{\ln a}{T} < \frac{1}{2}\sigma_{\mathbf{x}^*}^2 \quad .$$

Then, the minimal shortfall probability is smaller than 0.50 *and is attained at*

$$A_0 = 0 \quad , \quad m = \frac{1}{\sigma_{\mathbf{x}^*}} \sqrt{2\left(c + \frac{\ln a}{T}\right)}.$$

Proof. See Appendix IV . ■

Comment Since it is optimal to choose $A_0 = 0$ we are back in the framework of section 3.2 and the optimal leverage factor m corresponds to the result of proposition 2.

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Appendix I

Proof of Formula 4. Applying Itô's Lemma on

$$F_t = \frac{W_t}{L_t}$$

leads to

$$dF_t = \frac{dW_t}{L_t} - \frac{W_t}{L_t^2} dL_t + \frac{W_t}{L_t^3} (dL_t)^2 - \frac{1}{L_t^2} dW_t dL_t$$

or

$$\frac{dF_t}{F_t} = \frac{dW_t}{W_t} - \frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2 - \frac{dW_t}{W_t} \cdot \frac{dL_t}{L_t}.$$

Inserting (2), (3) implies

$$\frac{dF_t}{F_t} = \left[-c + \boldsymbol{\pi}^T \mathbf{x} - \boldsymbol{\pi}^T \mathbf{b} + \mathbf{b}^T \sigma\sigma^T \mathbf{b} - \mathbf{x}^T \sigma\sigma^T \mathbf{b}\right] dt + (\mathbf{x} - \mathbf{b})^T \sigma d\mathbf{Z}_t$$

which corresponds to (4). ■

Proof of Proposition 1 and Formulas 12 and 13. If (10) is satisfied then $g(\mathbf{y}) > 0$ holds for $\mathbf{y} = V^{-1}\boldsymbol{\pi} - \mathbf{b}$ and (8) can be applied. Therefore the minimal shortfall probability is smaller than 1. Since

$$\lim_{\|\mathbf{y}\| \to 0} \frac{g(\mathbf{y})}{[h(\mathbf{y})]^2} = -\infty \ , \ \lim_{\|\mathbf{y}\| \to \infty} \frac{g(\mathbf{y})}{[h(\mathbf{y})]^2} = -\frac{1}{2} \ , \ g\left(V^{-1}\boldsymbol{\pi} - \mathbf{b}\right) > 0$$

holds, (9) has a solution $\mathbf{y}^I \neq \mathbf{0}$.

From

$$\mathbf{grad} \frac{g(\mathbf{y}^I)}{[h(\mathbf{y}^I)]^2} = \mathbf{0}$$

one concludes that $\mathbf{y}^I$ must be of the form

$$\mathbf{y}^I = \lambda(V^{-1}\boldsymbol{\pi} - \mathbf{b}) = \lambda(\mathbf{x}^* - \mathbf{b}),$$

where λ has to be determined.

Since

$$\frac{g(\mathbf{y}^I)}{[h(\mathbf{y}^I)]^2} = \frac{-c + \left(\lambda - \frac{\lambda^2}{2}\right) \sigma^2_{\mathbf{x}^*,\mathbf{b}}}{\lambda^2 \sigma^2_{\mathbf{x}^*,\mathbf{b}}} = -\frac{c}{\lambda^2 \sigma^2_{\mathbf{x}^*,\mathbf{b}}} + \frac{1}{\lambda} - \frac{1}{2}$$

attains its maximum at

$$\lambda^I = \frac{2c}{\sigma^2_{\mathbf{x}^*,\mathbf{b}}}$$

the proof of Proposition 1 is complete.

Formulas (12) and (13) result from

$$\begin{aligned} \frac{g(\mathbf{y}^I)}{[h(\mathbf{y}^I)]^2} &= -\frac{\sigma^2_{\mathbf{x}^*,\mathbf{b}}}{4c} + \frac{\sigma^2_{\mathbf{x}^*,\mathbf{b}}}{2c} - \frac{1}{2} \\ &= \frac{\sigma^2_{\mathbf{x}^*,\mathbf{b}}}{4c} - \frac{1}{2} = \frac{\pi_{x^*}}{4c} - \frac{1}{2} \quad \text{for } \mathbf{b} = 0. \end{aligned}$$

■

Appendix II

Proof of Proposition 2 and Formula 17. If (15) is satisfied, then $z(\mathbf{y}) < 0$ holds for $\mathbf{y} = \mathbf{x}^* - \mathbf{b}$. Hence, the minimal shortfall probability is smaller than 0.5.

Since

$$\lim_{\|\mathbf{y}\|\to 0} \frac{\sqrt{T}}{h(\mathbf{y})} \cdot \left(\frac{\ln a}{T} - g(\mathbf{y})\right) = \infty \quad , \quad \lim_{\|\mathbf{y}\|\to \infty} \frac{\sqrt{T}}{h(\mathbf{y})} \cdot \left(\frac{\ln a}{T} - g(\mathbf{y})\right) = \infty$$

(14) has a solution $\mathbf{y}^{II} \neq 0$.

As in the proof of Proposition 1 one can show that $\mathbf{y}^{II}$ is of the form

$$\mathbf{y}^{II} = \lambda(\mathbf{x}^* - \mathbf{b}).$$

Moreover

$$\begin{aligned} \frac{\sqrt{T}}{h(\mathbf{y}^{II})} \cdot \left(\frac{\ln a}{T} - g(\mathbf{y}^{II})\right) &= \frac{\sqrt{T}}{\lambda \sigma_{\mathbf{x}^*,\mathbf{b}}} \cdot \left[\frac{\ln a}{T} + c - \left(\lambda - \frac{\lambda^2}{2}\right)\sigma^2_{\mathbf{x}^*,\mathbf{b}}\right] \\ &= \frac{\sqrt{T}}{\sigma_{\mathbf{x}^*,\mathbf{b}}} \cdot \left[\frac{1}{\lambda}\left(\frac{\ln a}{T} + c\right) - \sigma^2_{\mathbf{x}^*,\mathbf{b}} + \frac{\lambda}{2}\sigma^2_{\mathbf{x}^*,\mathbf{b}}\right] \end{aligned}$$

attains its minimum at

$$\lambda^{II} = \frac{1}{\sigma_{\mathbf{x}^*,\mathbf{b}}} \sqrt{2\left(\frac{\ln a}{T} + c\right)} \ .$$

This completes the proof of Proposition 2.

Formula (17) results from inserting λ^{II} into (14) which leads to

$$z\left(\mathbf{y}^{II}\right) = \sqrt{T}\left(\sqrt{2\left(\frac{\ln a}{T} + c\right)} - \sigma_{\mathbf{x}^*,\mathbf{b}}\right) .$$

■

Appendix III

Proof of Proposition 3. Since $c > 0$ we have

$$\lim_{\|\mathbf{y}\|\to\infty} \left(-c + \mathbf{y}^T(\boldsymbol{\pi} - V\mathbf{b}) - \frac{1}{2}\mathbf{y}^T V\mathbf{y} - \frac{|z_\alpha|}{\sqrt{T}} \cdot \left(\mathbf{y}^T V \mathbf{y}\right)^{0.5}\right) = -\infty.$$

Therefore (18) has a solution $\mathbf{y}^{III}$. By a similar argument as in the proof of Proposition 1 $\mathbf{y}^{III}$ is of the form

$$\mathbf{y}^{III} = \lambda\left(\mathbf{x}^* - \mathbf{b}\right) \quad , \ \lambda \geqslant 0 \ .$$

Inserting $\mathbf{y}^{III}$ into (18) leads to

$$-c + \lambda\sigma^2_{\mathbf{x}^*,\mathbf{b}} - \frac{\lambda^2}{2}\sigma^2_{\mathbf{x}^*,\mathbf{b}} - \lambda\frac{|z_\alpha|}{\sqrt{T}}\sigma_{\mathbf{x}^*,\mathbf{b}}.$$

The maximum is attained at

$$\mathbf{y}^{III} = \begin{cases} 0 & \text{, if } \; |z_\alpha| \geqslant \sqrt{T}\sigma_{\mathbf{x}^*,\mathbf{b}}, \\ 1 - \dfrac{|z_\alpha|}{\sqrt{T}\cdot\sigma_{\mathbf{x}^*,\mathbf{b}}}, & \text{if } \; |z_\alpha| < \sqrt{T}\sigma_{\mathbf{x}^*,\mathbf{b}}. \end{cases}$$

■

Appendix IV

Proof of Proposition 4. According to (20) a shortfall occurs if

$$e^{\left[-c+\left(m-\frac{m^2}{2}\right)\sigma^2_{\mathbf{x}^*,\mathbf{b}}\right]T+m\sigma_{\mathbf{x}^*,\mathbf{b}}\hat{Z}_T} < \frac{a-\dfrac{A_0}{W_0}e^{-cT}}{1-\dfrac{A_0}{W_0}}. \tag{21}$$

A minimal shortfall probability can only be attained if the right hand side of (21) is minimized. The assumption $c+\dfrac{\ln a}{T}>0$ implies $e^{-cT}<a$. Therefore the right hand side of (21) attains its minimum at $A_0=0$. Now Proposition 4 results as a special case of Proposition 2, that is when $\mathbf{b}=\mathbf{0}$. ■

Zusammenfassung

Eine natürliche Zielsetzung vieler institutioneller und privater Investoren besteht darin, langfristig gegenüber einer Referenzstrategie (z.B. Geldmarktanlage, Bondindex, etc.) eine Zusatzrendite zu erzielen und gleichzeitig die Wahrscheinlichkeit eines Shortfalls minimal zu halten. In der vorliegenden Arbeit wird dargelegt, dass eine substantielle Zusatzrendite auch langfristig hohe Shortfall-Wahrscheinlichkeiten bedingt.

Im Modell folgen die Kurse der Anlagemöglichkeiten einer geometrischen Brownschen Bewegung. Es werden zwei Typen von Shortfall unterschieden. Ein Shortfall vom Typ I tritt auf, falls das Anlageziel in irgendeinem Zeitpunkt um einen festen Prozentsatz verfehlt wird. Ein Shortfall vom Typ II liegt vor, falls das Anlageziel am Ende der Planungsperiode nicht erreicht wird. Zuerst werden nur zeitlich konstante Portfoliogewichte zugelassen. Für beide Typen lässt sich zeigen, dass die Minimierung der Shortfall-Wahrscheinlichkeiten bei vorgegebener Zusatzrendite äquivalent zum Merton-Problem ist. Moderate Shortfall-Wahrscheinlichkeiten resultieren unter realitätsnahen Parameterwerten nur bei sehr geringen Zusatzrenditen. Schliesslich wird nachgewiesen, dass der Einsatz von "Constant Proportion Portfolio Insurance" (CPPI) zu keiner Reduktion der Shortfall-Wahrscheinlichkeit führt.